

Relativistic entanglement in single-particle quantum states using Non-Linear entanglement witnesses

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Abstract. In this study, the spin-momentum correlation of one massive spin- $\frac{1}{2}$ and spin-1 particle states, which are made based on projection of a relativistic spin operator into timelike direction is investigated. It is shown that by using Non-Linear entanglement witnesses (NLEWs), the effect of Lorentz transformation would decrease both the amount and the region of entanglement.

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1. Introduction:

Relativistic aspects of quantum entanglement, especially in Einstein, Podolsky and Rosen (EPR) correlations [1] and the Bell inequality [2, 3, 4, 5, 6, 7], have recently attracted much attention. The quantum states in inertial frame of reference transform under Wigner rotation [8] via Non-Abelian continuous group $SU(2)$ representation in momentum space. Therefore, under Lorentz transformation, spin becomes entangled with the particle's momentum. Photon polarization qubits behave similarly and linear polarization states of photon will be seen in moving frame in the form of depolarized states [9].

Other aspects of relativistic entanglement are based on spinor formalism, which is designed for four-dimensional space-time [10]. In Refs.[11, 12, 13, 14, 15], Czachor and co-worker have used spinor method and shown that the *helicity*-momentum states, which are the irreducible representation of Poincare group, are the projection of the relativistic spin operator on timelike direction in momentum representation.

Separability of the quantum states and quantification of entanglement in composite systems are perhaps the most important features of quantum information. The first related criterion for distinguishing entangled states from separable ones, is the positive partial transpose (PPT) criterion, introduced by Peres [16]. Nonetheless, the strongest manner to characterize entanglement is using entanglement witnesses EWs [17, 18]. An EW is an observable \mathcal{W} whose expectation value is nonnegative on any separable state, but strictly negative on an entangled state. Recently, there has been an increased interest in the NLEWs because of their improved detection with respect to linear EWs. An NLEW is any bound on nonlinear function of observables which is satisfied by separable states but violated by some entangled states [19, 20]. For the first time in Refs.[22, 23], it was actually shown that EW can be expressed as a measure of entanglement. However, the main objective of this paper is to show that NLEWs can also be very helpful to the quantification of entanglement.

In our previous work [24], we studied spin-momentum correlation in single-particle spin- $\frac{1}{2}$ quantum states by using concurrence and showed that the amount of spin-momentum correlation depends on the angle between spin and momentum. But here, We quantify the spin-momentum correlation for a massive spin half and one relativistic single-particle in $2 \otimes 2$ (as two-qubit system) and $2 \otimes 3$ dimension Hilbert space using the NLEWs. For simplicity, instead of superposition of momenta, we have used two momentum eigenstates (p_1 and p_2) and in 2D momentum subspace, our results suggest the effect of the Lorentz transformation would decrease both the amount and the region of entanglement.

This paper is organized as follows: Sec. II, is devoted to single-particle spin half and one quantum states. In Sec. III the spin-momentum correlation is calculated in relativistic single-particle spin half and one mixed states by using the NLEW. Finally, in Sec. IV the results are summarized and conclusions are presented.

2. Relativistic one massive spin- $\frac{1}{2}$ and one particle quantum states in *helicity* basis

2.1. Unitary representation of Poincare group

The Poincare group $\text{ISO}(3,1)$ is a semidirect product of $\text{SO}(3,1)$ group of Lorentz transformation with the following generators

$$P^\mu, J^{\mu\nu} \quad \mu, \nu = 0, 1, 2, 3 \quad (1)$$

where $P^\mu, \mu = 1, 2, 3$ are components of momentum operators, and $P^0 = p^0 = \sqrt{m^2 c^2 + \mathbf{p}^2}$ is *Hamiltonian*, and $\mathbf{J} = \{J^{23}, J^{31}, J^{12}\}$ are components of angular momentum and $\mathbf{K} = \{J^{10}, J^{20}, J^{30}\}$ are boost operators. Likewise, unitary representation of Poincare group can be written in the following form:

$$\mathbf{K} = -ip^0 \nabla_{\mathbf{p}} - \frac{\mathbf{p} \times \mathbf{s}}{mc + p^0}, \quad (2a)$$

$$\mathbf{J} = -i\mathbf{p} \times \nabla_{\mathbf{p}} + \mathbf{s}, \quad (2b)$$

$$\mathbf{P} = \mathbf{p}, \quad (2c)$$

where $\nabla_{\mathbf{p}}$ is the partial differential as $\frac{\partial}{\partial \mathbf{p}}$ and \mathbf{s} denotes the finite dimensional angular momentum corresponding to $\mathbf{s}^2 = s(s+1)$. We know that the Poincare group is of rank 2, so there are only two independent Casimir invariant operators, which are squared mass and the square of the Pauli-Lubanski vector W_μ that commute with all generators of the algebra. Let's use this definition to write down the form of Casimir's operators.

$$P^2 = P_\mu P^\mu, \quad W^2 = W_\mu W^\mu, \quad (3)$$

the Pauli-Lubanski vector can be written as:

$$W_\mu = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} J^{\nu\rho} P^\sigma \quad (4)$$

where $\epsilon_{\mu\nu\rho\sigma}$ is anti symmetry Levi-Civita tensor and takes the value +1 if $\mu\nu\rho\sigma$ is an even permutation, -1 if it is an odd permutation and zero otherwise ($\epsilon_{0123} = -\epsilon^{0123} = 1$). It is common to denote the Pauli-Lubanski vector as

$$W^\mu = (W^0, \vec{\mathbf{W}}) = (\mathbf{P} \cdot \mathbf{J}, P_0 \mathbf{J} - \mathbf{P} \times \mathbf{K}). \quad (5)$$

We can define the popular center-of-mass position operator as

$$\mathbf{Q} = \frac{-1}{2} \left[\frac{1}{P_0} \mathbf{K} + \mathbf{K} \frac{1}{P_0} \right]. \quad (6)$$

Then, the orbital angular momentum is

$$\mathbf{L} = \mathbf{Q} \times \mathbf{P}. \quad (7)$$

Therefore, We can write a relativistic spin operator in the following way

$$\mathbf{S} = \mathbf{J} - \mathbf{L}, \quad (8)$$

and using the Pauli-Lubanski vector, we get

$$\mathbf{S} = \frac{\mathbf{W}}{P^0}. \quad (9)$$

So, the spacelike component of Pauli-Lubanski vector is proportional to the relativistic spin operator. If we choose the timelike vector $b^a = (b^0, 0, 0, 0)$, then after projection of Pauli-Lubanski vector on it, we obtain

$$b^a W_a = b^0 W^0 = b^0 \mathbf{p} \cdot \mathbf{J}, \quad (10)$$

which is helicity. On the other hand, the projection of Eq.(5) onto the vector $b^a = (0, \mathbf{b})$ have eigenvalues as

$$\lambda_{\pm} = \frac{\pm 1}{2} \sqrt{(\mathbf{p} \cdot \mathbf{b})^2 + m^2 b^2}. \quad (11)$$

If the world-vector b^a could be a null tetrad, i.e., $b^2 = b^a b_a = 0$, we get

$$\lambda_{\pm} = \pm \frac{1}{2} (p \cdot b). \quad (12)$$

In momentum representation, we can write two energy-momentum world-vectors as a linear combination of the null tetrad and after projection of RSO onto the null tetrad (see Eq. (12)), we get the following eigenvalue which is similar to eigenvalue Pauli's matrix, i.e., σ_z (for details see [11]).

$$\lambda_{\pm} = \pm \frac{1}{2}. \quad (13)$$

2.2. single-particle spin- $\frac{1}{2}$ quantum state

Here in this paper, quantum state is made up of a single-particle having two types of degrees of freedom : momentum p and spin σ . The former is a continuous variable with Hilbert space of infinite dimension while the latter is a discrete one with Hilbert space of finite dimension. For simplicity, instead of using the superposition of momenta, we use only two momentum eigenstates (p_1 and p_2). However, we restrict ourselves to 2D momentum subspace with two eigenstates p_1 and p_2 , so the pure quantum state of such a system can always be written as

$$|\psi\rangle = \sum_{i=1}^2 \sum_{\sigma=0,1} c_{ij} |p_i, \sigma\rangle, \quad (14)$$

where $|p_{1(2)}\rangle$ and $|\sigma\rangle$ are eigenstates of momentum and spin operators, respectively. c_{ij} are complex coefficients such that $\sum_{i,j} |c_{ij}|^2 = 1$.

A bipartite quantum mixed state is defined as a convex combination of bipartite pure states (14), i.e.,

$$\rho_{\frac{1}{2}} = \sum_{i=1}^4 P_i |\psi_i\rangle \langle \psi_i|, \quad (15)$$

where $P_i \geq 0$, $\sum_i P_i = 1$. The subscript “ $\frac{1}{2}$ ” refers to the spin- $\frac{1}{2}$ and $|\psi_i\rangle$ ($i = 1, 2, 3, 4$) as four orthogonal maximal entangled Bell states (BD) belong to the product space $\mathcal{H}_p \otimes \mathcal{H}_\sigma$ and are well-known as

$$|\psi_1\rangle = \frac{1}{\sqrt{2}} (|p_1, 0\rangle + |p_2, 1\rangle), \quad (16a)$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(|p_1, 0\rangle - |p_2, 1\rangle), \quad (16b)$$

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}(|p_2, 0\rangle + |p_1, 1\rangle), \quad (16c)$$

$$|\psi_4\rangle = \frac{1}{\sqrt{2}}(|p_2, 0\rangle - |p_1, 1\rangle). \quad (16d)$$

in terms of momentum and spin states. The kets $|0\rangle$ and $|1\rangle$ are the eigenvectors of spin operator σ_z . We assumed that spin and momentum are parallel in the z-direction and in this case, the single-particle spin- $\frac{1}{2}$ state can be considered as a two-qubit system. For an observer in another reference frame S' , is described by an arbitrary boost Λ in the x-direction. The transformed BD states are given by (see Appendix A),

$$|\psi_i\rangle \longrightarrow U(\Lambda)|\psi_i\rangle = \sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda, p)) |\psi_i\rangle, \quad (17)$$

where $U(\Lambda)$ is a unitary representation of Lorentz transformation. It can be calculated that $|\psi_i\rangle$ will be orthogonal after Lorentz transformation, i. e.,

$$\langle \Lambda \psi_i | \Lambda \psi_j \rangle = \delta_{ij}. \quad (18)$$

The BD density matrix (15), which describes the state of the single-particle at non-relativistic frame, is exchanged to the density matrix $\rho_{\frac{1}{2}}^\Lambda$ after Lorentz transformation, i.e.,

$$\rho_{\frac{1}{2}} \longrightarrow U(\Lambda) \rho_{\frac{1}{2}} U(\Lambda)^\dagger, \quad (19)$$

therefore

$$\rho_{\frac{1}{2}}^\Lambda = U(\Lambda) \rho_{\frac{1}{2}} U(\Lambda)^\dagger = \sum_{i=1}^4 P_i |\Lambda \psi_i\rangle \langle \Lambda \psi_i|. \quad (20)$$

2.3. single-particle spin-1 quantum state

We assumed that for spin one, the particle moves with two momentum eigen-state p_1 and p_2 along the y-axis. For an observer in another reference frame S' described by an arbitrary boost Λ given by the velocity \vec{v}' in the z-direction, we have

$$\mathbf{p}_{1(2)} = (0, \pm p_{1(2)}, 0, E) = (0, m\sqrt{\gamma'_{1(2)}{}^2 - 1}, 0, m\gamma'_{1(2)}), \quad (21)$$

where

$$E^2 - p_{1(2)}^2 = m^2, \gamma'_{1(2)} = \sqrt{\frac{p_{1(2)}^2 + m^2}{m^2}}, \bar{\gamma}' = \frac{1}{\sqrt{1 - \beta'^2}}, \beta = \frac{v'}{c}. \quad (22)$$

According to Ref.[25], the Wigner representation for spin j is calculated as follows:

$$\mathcal{D}_{\sigma', \sigma}^j(W(\Lambda, p)) = e^{\frac{i\theta}{2}(\delta_{\sigma', \sigma+1}\sqrt{(j-\sigma)(j+\sigma+1)} + \delta_{\sigma', \sigma-1}\sqrt{(j+\sigma)(j-\sigma+1)})}, \quad (23)$$

where θ is angle around the x-axis. After some mathematical manipulations for spin one we get

$$\mathcal{D}_{\sigma',\sigma}^{(j=1)}(W(\Lambda, p_{1(2)})) = \begin{pmatrix} \frac{\cos \theta_{1(2)} + 1}{2} & \frac{i \sin \theta_{1(2)}}{\sqrt{2}} & \frac{\cos \theta_{1(2)} - 1}{2} \\ \frac{i \sin \theta_{1(2)}}{\sqrt{2}} & \cos \theta_{1(2)} & \frac{i \sin \theta_{1(2)}}{\sqrt{2}} \\ \frac{\cos \theta_{1(2)} - 1}{2} & \frac{i \sin \theta_{1(2)}}{\sqrt{2}} & \frac{\cos \theta_{1(2)} + 1}{2} \end{pmatrix}, \quad (24)$$

for simplicity, assume that $p_1 = -p_2 = p$, then $\theta_1 = -\theta_2 = \theta$.

In order to consider single spin one particle mixed state under Lorentz transformation we define the following density matrix in the rest frame as follows:

$$\rho_1 = x|\psi_5\rangle\langle\psi_5| + y|\psi_6\rangle\langle\psi_6| + \frac{1-x-y}{6}I, x+y \leq 1, \quad (25)$$

The subscript “1” refers to the spin one and $|\psi_{5(6)}\rangle$ are maximally entangled pure states, given by

$$|\psi_5\rangle = \frac{1}{\sqrt{2}}(|p_1, -1\rangle + |p_2, 1\rangle), \quad (26a)$$

$$|\psi_6\rangle = \frac{1}{\sqrt{2}}(|p_1, -1\rangle - |p_2, 1\rangle), \quad (26b)$$

where $|p_{1(2)}\rangle$ are two momentum eigen states of particle and $\{|\pm 1\rangle, |0\rangle\}$ are the three component spin ones as

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, |0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, |-1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (27)$$

By using (24), we can obtain the relativistic density matrix (25) as

$$\rho \longrightarrow \rho_1^\Lambda = U(\Lambda)\rho_1 U^\dagger(\Lambda). \quad (28)$$

3. Measure of entanglement of single-particle states using NLEW

3.1. Entanglement Witnesses

An entanglement witness \mathcal{W} is an observable that reveals the entanglement of some entangled state ρ , i.e., \mathcal{W} is such that $Tr(\mathcal{W} \cdot \rho_s) \geq 0$ for all separable ρ_s , but $Tr(\mathcal{W} \cdot \rho_{en}) < 0$. The existence of an EW for any entangled state is a direct consequence of Hahn-Banach theorem [26] and the space of separable density operators is convex and closed. Geometrically, EWs can be viewed as hyper planes that separate some entangled states from the set of separable states and, hyper plane indicated as a line corresponds to the state with $Tr[\mathcal{W} \cdot \rho_s] = 0$. According to Refs.[27, 28], an EW will be optimal if, for all positive operators P and $\epsilon > 0$, the operator

$$\mathcal{W}' = (1 + \epsilon)\mathcal{W} - \epsilon P, \quad (29)$$

is not an EW.

When talking about EWs, one has to take an important point into consideration: the so-called decomposable EWs (DEW), which can be written as

$$\mathcal{W} = \mathcal{P} + Q_1^{TA}, \quad \mathcal{P}, Q_1 \geq 0, \quad (30)$$

where the operator Q_1 is positive semidefinite. It can be easily verified that such witnesses cannot detect any bound entangled states. \mathcal{W} is non-decomposable EW if it can not be put in the form (30) (for more details see [29]). One should notice that only non-decomposable EWs can detect PPT entangled states.

3.2. Measure of entanglement of single spin- $\frac{1}{2}$ particle using NLEW

According to Ref. [30, 15] we present an NLEW for a bipartite system $\mathcal{H}_2 \otimes \mathcal{H}_2$ as follows

$$\mathcal{W}_{2 \otimes 2} = I_2 \otimes I_2 + \sum_{i,j=1}^4 A_{i,j} \hat{\mathcal{O}}_i \otimes \hat{\mathcal{O}}_j, \quad (31)$$

where I_2 is a 2×2 identity matrix, $A_{i,j}$ are some parameters, and \mathcal{O}_i s are Hermitian operators from first (second) party Hilbert space as following

$$\mathcal{O}_1 = |0\rangle\langle 0|, \quad (32a)$$

$$\mathcal{O}_2 = |1\rangle\langle 1|, \quad (32b)$$

$$\mathcal{O}_3 = \frac{1}{\sqrt{2}}(\mathcal{O}_1 + \mathcal{O}_2), \quad (32c)$$

$$\mathcal{O}_4 = \frac{-i}{\sqrt{2}}(\mathcal{O}_1 - \mathcal{O}_2). \quad (32d)$$

We introduce the maps for any separable state ρ_s which map the convex set of separable states to a bounded convex region named as feasible region (FR). Then, recalling the definition of an EW, we imposed the first condition which is the problem of the minimization of expectation values of witness operators \mathcal{W} with respect to separable states, i.e.,

$$\min Tr(\mathcal{W} \cdot \rho_s) \geq 0, \quad (33)$$

In the second step, for a given ρ_{ent} , we imposed the second condition for an EW, $Tr(\mathcal{W} \cdot \rho_{ent}) < 0$. Now the objective function (which will be minimized) is $Tr(\mathcal{W} \cdot \rho_{ent})$, and the inequality constraints come from the first step solution. So, this problem can be written as a convex optimization problem. As mentioned above, we will use two steps towards finding the parameters $\mathcal{A}_{i,j}$ for the density matrix and fully characterize NLEWs based on exact convex optimization method for single spin- $\frac{1}{2}$ quantum mixed state. Now we define $\tilde{\rho}_{\frac{1}{2}}$ as follows:

$$\tilde{\rho}_{\frac{1}{2}i,j} = Tr(\rho_{\frac{1}{2}} \cdot \mathcal{O}_i \otimes \mathcal{O}_j), \quad (34)$$

Obviously, these components are associated with components of witness matrix, i.e., $\mathcal{A}_{\mu,\nu}$. If matrix Z is a symmetric matrix (i.e. $Z^t = Z$) then $\tilde{\rho}_{\frac{1}{2}} = -2\mathcal{A}Z$ or

$$Z = \frac{1}{2}\mathcal{A}^t\tilde{\rho}_{\frac{1}{2}} = \frac{1}{2}(\tilde{\rho}_{\frac{1}{2}}^t\tilde{\rho}_{\frac{1}{2}})^{\frac{1}{2}}, \quad (35)$$

where

$$\mathcal{A} = -\frac{1}{2}\tilde{\rho}_{\frac{1}{2}}Z^{-1}. \quad (36)$$

So, after some calculation we get

$$A_{11} = A_{22} = \frac{1}{2}\left(\frac{-P_1 - P_2 + P_3 + P_4}{\sqrt{(P_1 + P_2 - P_3 - P_4)^2}} - 1\right), \quad (37a)$$

$$A_{12} = A_{21} = \frac{1}{2}\left(\frac{P_1 + P_2 - P_3 - P_4}{\sqrt{(P_1 + P_2 - P_3 - P_4)^2}} - 1\right), \quad (37b)$$

$$A_{33} = \frac{-P_1 + P_2 - P_3 + P_4}{\sqrt{(P_1 - P_2 + P_3 - P_4)^2}}, \quad (37c)$$

$$A_{44} = \frac{P_1 - P_2 - P_3 + P_4}{\sqrt{(P_1 - P_2 - P_3 + P_4)^2}}, \quad (37d)$$

and other parameters are zero. Then, by using (49) the NLEW can be written as

$$\begin{aligned} \mathcal{W}_{2\otimes 2} = & I_2 \otimes I_2 + A_{11}(\mathcal{O}_1 \otimes \mathcal{O}_1 + \mathcal{O}_2 \otimes \mathcal{O}_2) + A_{12}(\mathcal{O}_1 \otimes \mathcal{O}_2 + \mathcal{O}_2 \otimes \mathcal{O}_1) \\ & + A_{33}\mathcal{O}_3 \otimes \mathcal{O}_3 + A_{44}\mathcal{O}_4 \otimes \mathcal{O}_4. \end{aligned} \quad (38)$$

Then measuring the observable $\mathcal{W}_{2\otimes 2}$ of $\rho_{\frac{1}{2}}$ gives a good estimate of the expected value of $\mathcal{W}_{2\otimes 2}$,

$$Tr[\mathcal{W}_{2\otimes 2} \cdot \rho_{\frac{1}{2}}] = \frac{1}{2}\{1 - |P_1 + P_2 - P_3 - P_4| - |P_1 - P_2 + P_3 - P_4| - |P_1 - P_2 - P_3 + P_4|\},$$

using $\sum_i P_i = 1$ we obtain

$$Tr[\mathcal{W}_{2\otimes 2} \cdot \rho_{\frac{1}{2}}] = (1 - 2P_i). \quad (39)$$

We use the concurrence to measure the entanglement between spin and momentum of single particle. It is defined as

$$C(\rho) = \max(0, \nu_1 - \nu_2 - \nu_3 - \nu_4),$$

where the quantities $\nu_1 \geq \nu_2 \geq \nu_3 \geq \nu_4$ in decreasing order are the square roots of the eigenvalues of the matrix

$$R = \rho(\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y),$$

for Bell-diagonal density matrix of (15) after some calculation we get

$$C(\rho_{\frac{1}{2}}) = (2P_i - 1), \quad (40)$$

and Eq.(39), we obtain

$$C(\rho_{\frac{1}{2}}) = -Tr[\mathcal{W}_{2\otimes 2} \cdot \rho_{\frac{1}{2}}], \quad (41)$$

where $Tr[\mathcal{W}_{2\otimes 2} \cdot \rho_{\frac{1}{2}}]$ is zero when $P_i = \frac{1}{2}$ and exactly coincides with the concurrence of two-qubit Bell-diagonal mixed state (15).

Moreover, we have made the NLEW for relativistic density matrix (20) and taking the trace we obtain:

$$Tr[\mathcal{W}_{2\otimes 2} \cdot \rho_{\frac{1}{2}}^\Lambda] = A - \frac{1}{4}(\sqrt{B - 4\sqrt{C}} + \sqrt{B + 4\sqrt{C}}), \quad (42)$$

where

$$A = 1 - \frac{1}{2}\{\sqrt{(P_1 + P_2 - P_3 - P_4)^2} + \sqrt{(P_1 - P_2 - P_3 + P_4)^2}\} \cos \Omega_p, \quad (43a)$$

$$B = 8(P_1 P_3 + P_2 P_4) \cos^2 \Omega_p - 2(P_1^2 + P_2^2 + P_3^2 + P_4^2)(-3 + \cos 2\Omega_p), \quad (43b)$$

$$C = 4(P_1 + P_3)^2(P_2 + P_4)^2 + 4((P_1^2 - P_3^2)^2 + (P_2^2 + P_4^2)^2) \sin^2 \Omega_p + 4(P_1 - P_3)^2(P_2 - P_4)^2 \sin^4 2\Omega_p. \quad (43c)$$

We suppose that the Wigner angle Ω_p is defined by (see Appendix A)

$$\Omega_{p_1} = \Omega_{p_2} = \Omega_p. \quad (44)$$

After some calculations and using (39), we can obtain

$$Tr[\mathcal{W}_{2\otimes 2} \cdot \rho_{\frac{1}{2}}] \leq Tr[\mathcal{W}_{2\otimes 2} \cdot \rho_{\frac{1}{2}}^\Lambda] \Rightarrow C(\rho_{\frac{1}{2}}(\Omega_p = 0)) \geq C(\rho_{\frac{1}{2}}^\Lambda(\Omega_p)). \quad (45)$$

This result indicates that both the amount and region of entanglement decrease (see Figure.2).

3.3. Measure of entanglement of single-particle spin-one mixed state using NLEW

We know that positive partial transpose (PPT) criterion is a necessary and sufficient condition for determining entangled states living in Hilbert spaces $H_2 \otimes H_2$ and $H_2 \otimes H_3$. But, with using the NLEW we want to measure of entanglement of bipartite system in $H_2 \otimes H_3$ Hilbert space.

In this section, we want to calculate the geometric measure of entanglement of single spin one relativistic particle mixed states in the $2 \otimes 3$ Hilbert space which are spanned by the following basis vectors

$$\{|p_{1(2)}\rangle \otimes |\pm 1\rangle, |p_{1(2)}\rangle \otimes |0\rangle\}, \quad (46)$$

We consider the density matrix of (25) and obtain the FR by imposing PPT conditions with respect to each party. The PPT condition was applied on the first party and the eigenvalues of ρ^T in the rest frame (i.e., $\theta = 0$) are given by

$$\lambda_1 = \frac{1}{6}(1 + 2x - 4y), \quad (47a)$$

$$\lambda_2 = \frac{1}{6}(1 - x - y), \quad (47b)$$

$$\lambda_3 = \frac{1}{6}(1 - 4x + 2y), \quad (47c)$$

$$\lambda_4 = \frac{1}{6}(1 + 2x + 2y), \quad (47d)$$

The positivity of density matrix (25) imposes the following constraints on the parameters

$$\lambda_1 = \frac{1}{6}(1 - x - y), \quad (48a)$$

$$\lambda_2 = \frac{1}{6}(1 + 5x - y), \quad (48b)$$

$$\lambda_3 = \frac{1}{6}(1 - x + 5y), \quad (48c)$$

we present an EW for mixed state (25) in $\mathcal{H}_2 \otimes \mathcal{H}_3$ Hilbert space as follows

$$\mathcal{W}_{2 \otimes 3} = I_2 \otimes I_3 + \sum_{i=1}^4 \sum_{j=1}^9 \mathcal{A}_{i,j} \hat{\mathcal{O}}_i \otimes \hat{\mathcal{Q}}_j, \quad (49)$$

where I_2 and I_3 are 2×2 and 3×3 identity matrix, respectively. $\mathcal{A}_{i,j}$ s are some parameters, and \mathcal{O}_i s which have been defined in the previous section (see 32a until 32d) and \mathcal{Q}_j s are Hermitian operators from second party Hilbert space as following

$$\mathcal{Q}_1 = |1\rangle\langle 1|, \quad \mathcal{Q}_2 = |0\rangle\langle 0|, \quad \mathcal{Q}_3 = |-1\rangle\langle -1|, \quad (50a)$$

$$\mathcal{Q}_4 = \frac{1}{\sqrt{2}}(\mathcal{Q}_1 + \mathcal{Q}_2), \quad \mathcal{Q}_5 = \frac{1}{\sqrt{2}}(\mathcal{Q}_1 + \mathcal{Q}_3), \quad \mathcal{Q}_6 = \frac{1}{\sqrt{2}}(\mathcal{Q}_2 + \mathcal{Q}_3), \quad (50b)$$

$$\mathcal{Q}_7 = \frac{i}{\sqrt{2}}(\mathcal{Q}_1 - \mathcal{Q}_2), \quad \mathcal{Q}_8 = \frac{i}{\sqrt{2}}(\mathcal{Q}_1 - \mathcal{Q}_3), \quad \mathcal{Q}_9 = \frac{i}{\sqrt{2}}(\mathcal{Q}_2 - \mathcal{Q}_3). \quad (50c)$$

So, after some mathematical manipulations we get

$$\mathcal{A}_{11} = \mathcal{A}_{23} = \frac{-2\sqrt{3} - (\sqrt{3(x+y)^2} - 3\sqrt{2 + (x+y)^2})}{6\sqrt{2 + (x+y)^2}}, \quad (51a)$$

$$\mathcal{A}_{12} = \mathcal{A}_{22} = \frac{-1 + x + y}{\sqrt{6 + 3(x+y)^2}}, \quad (51b)$$

$$\mathcal{A}_{13} = \mathcal{A}_{21} = \frac{-2\sqrt{3} - (\sqrt{3(x+y)^2} + 3\sqrt{2+(x+y)^2})}{6\sqrt{2+(x+y)^2}}, \quad (51c)$$

$$\mathcal{A}_{35} = \mathcal{A}_{48} = -1. \quad (51d)$$

Obviously, we can see that the $\mathcal{A}^\dagger \mathcal{A} = \mathcal{A} \mathcal{A}^\dagger \leq 1$. So we have an EW candidate as

$$\begin{aligned} \mathcal{W}_{2\otimes 3} = & I_2 \otimes I_3 + \mathcal{A}_{11}(\mathcal{O}_1 \otimes \mathcal{Q}_1 + \mathcal{O}_2 \otimes \mathcal{Q}_3) + \mathcal{A}_{12}(\mathcal{O}_1 \otimes \mathcal{Q}_2 + \mathcal{O}_2 \otimes \mathcal{Q}_2) + \\ & \mathcal{A}_{13}(\mathcal{O}_1 \otimes \mathcal{Q}_3 + \mathcal{O}_2 \otimes \mathcal{Q}_1) - (\mathcal{O}_3 \otimes \mathcal{Q}_5 + \mathcal{O}_4 \otimes \mathcal{Q}_8). \end{aligned} \quad (52)$$

Its expectation values in separable states are all nonnegative while its expectation value in the state (25) reads

$$Tr(\mathcal{W}_{2\otimes 3} \cdot \rho_1) = \frac{1}{6} \{6 - 6\sqrt{(x-y)^2} - 3\sqrt{(x+y)^2} - \sqrt{6+3(x+y)^2}\}. \quad (53)$$

We want to show that for some values of the parameters x and y , the amount of entanglement of mixed state (25) changes. To this aim, we have constructed a non-linear EW (52). For convenience, we consider point $x = 0, y = 1$ which is entangled, i.e., $Tr(\mathcal{W}_{2\otimes 3} \cdot \rho) = -1$. So, in another inertial frame S' that moves with velocity v with respect to rest frame with $\theta = \frac{\pi}{4}$, we have

$$\tau_1 = \frac{1}{6}(1 - x - y), \quad (54a)$$

$$\tau_{2(3)} = \frac{1}{12}(2 - 2x + 4y \pm 3\sqrt{3x^2 + y^2}), \quad (54b)$$

$$\tau_{3(4)} = \frac{1}{12}(2 + 4x - 2y \pm 3\sqrt{x^2 + 3y^2}), \quad (54c)$$

where τ_i 's are eigenvalues of the partial transpose $\rho_1^\Lambda = U(\Lambda)\rho_1 U(\Lambda)^\dagger$. So, the non-linear EW, $\mathcal{W}_{2\otimes 3}$, can be constructed in a similar way for relativistic density matrix. Its expectation value with respect to relativistic density matrix ρ_1^Λ is given by

$$Tr(\mathcal{W}_{2\otimes 3} \cdot \rho_1^\Lambda) = 1 - \frac{\sqrt{3}}{4} \{ \sqrt{(x-y)^2} + \sqrt{(x+y)^2} + \sqrt{B - 2\sqrt{A}} + \sqrt{B + 2\sqrt{A}} \}, \quad (55)$$

where

$$B = 4 + 11x^2 - 8xy + 11y^2, \quad (56a)$$

$$\begin{aligned} A = & 4 + 4x^2(1+x)(5x-1) - 4xy(4x+11x^2-9) + \\ & y^2(97x^2-16x-4) + 4(4-11x)y^3 + 20y^4. \end{aligned} \quad (56b)$$

After some calculations one arrives at

$$Tr[\mathcal{W}_{2\otimes 3} \cdot \rho_1] \leq Tr[\mathcal{W}_{2\otimes 3} \cdot \rho_1^\Lambda]. \quad (57)$$

For example, for $x = 0, y = 1$ we obtain $Tr(\mathcal{W} \cdot \rho_1^\Lambda) = \frac{-\sqrt{3}}{2}$, this result shows that the measure of entanglement single spin one particle decreases when the velocity of the

observer increases. In Fig. 1, we have shown that the effect of the Lorentz transformation is to increase the region of separable states.(see Figure.1)

Moreover, using the geometric measure of entanglement for quantum systems defined in [21, 22], we consider entangled state (25) with $x = 0$ and $y = 1$ from the nearest separable state in the case $x = \frac{1}{2}, y = \frac{1}{2}$, which have been obtained using the convex optimization. We have

$$\|\rho - \rho_s\|^2 = \frac{\cos^2 \theta}{\sqrt{2}}, \quad (58)$$

and

$$Tr[\mathcal{W} \cdot \rho] = \frac{-1}{2} \cos \theta \sqrt{6 - 2 \cos 2\theta}, \quad (59)$$

which is separable in case of $\theta = \frac{\pi}{2}$.

4. SUMMARY AND CONCLUSION

In this paper, we have considered spin-momentum correlation of single spin half and one relativistic particle quantum states by using the NLEW, which in case of spin half coincides exactly with the concurrence. Then, we have shown that for single-particle quantum states which have been constructed based on two types of degrees of freedom spin and momentum, both the amount and the region of entanglement between spin and momentum decrease under Lorentz transformation, with respect to the increasing of observer velocity. Likewise, we have obtained the nearest distance bound of separable states from entangled density matrix and shown that it leads to zero in the ultrarelativistic limit. Finally, a natural question arises as how the previous results would generalize to the case of other mixed states. So, the calculations in this study are intended as a point of reference for the development of an understanding of the measure of entanglement in critical quantum systems based on NLEW.

APPENDIX A

Wigner representation for spin- $\frac{1}{2}$

In Ref. [25], it is shown that the effect of an arbitrary Lorentz transformation Λ unitarily implemented as $U(\Lambda)$ on single-particle states is

$$U(\Lambda)(|p\rangle \otimes |\sigma\rangle) = \sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda, p))(|\Lambda p\rangle \otimes |\sigma'\rangle),$$

where

$$W(\Lambda, p) = L^{-1}(\Lambda p)\Lambda L(p),$$

is the Wigner rotation [8]. We can view this Wigner rotation as follows: we perform the Lorentz transformation $L(p)$ on the rest frame to obtain a moving frame 1, followed by a transformation from frame 1 to frame 2 with Λ . Then we return to the rest frame

by further performing $L^{-1}(\Lambda p)$. This rotation of the local frame of rest is the kinematic effect that causes the Thomas precession. We will consider two reference frames in this work: one is the rest frame S and the other is the moving frame S' in which a particle whose four-momentum p in S is seen as boosted with the velocity \vec{v} . By setting the boost and particle moving directions in the rest frame to be \hat{v} with \hat{e} as the normal vector in the boost direction and \hat{p} , respectively, and $\hat{n} = \hat{e} \times \hat{p}$, the Wigner representation for spin- $\frac{1}{2}$ particle is found as [7],

$$D^{\frac{1}{2}}(W(\Lambda, p)) = \cos \frac{\Omega_{\vec{p}}}{2} + i \sin \frac{\Omega_{\vec{p}}}{2} (\vec{\sigma} \cdot \hat{n}),$$

where

$$\begin{aligned} \cos \frac{\Omega_{\vec{p}}}{2} &= \frac{\cosh \frac{\alpha}{2} \cosh \frac{\delta}{2} + \sinh \frac{\alpha}{2} \sinh \frac{\delta}{2} (\hat{e} \cdot \hat{p})}{\sqrt{[\frac{1}{2} + \frac{1}{2} \cosh \alpha \cosh \delta + \frac{1}{2} \sinh \alpha \sinh \delta (\hat{e} \cdot \hat{p})]}} \\ \sin \frac{\Omega_{\vec{p}}}{2} \hat{n} &= \frac{\sinh \frac{\alpha}{2} \sinh \frac{\delta}{2} (\hat{e} \times \hat{p})}{\sqrt{[\frac{1}{2} + \frac{1}{2} \cosh \alpha \cosh \delta + \frac{1}{2} \sinh \alpha \sinh \delta (\hat{e} \cdot \hat{p})]}} \end{aligned}$$

and

$$\cosh \alpha = \gamma = \frac{1}{\sqrt{1 - \beta^2}}, \cosh \delta = \frac{E}{m}, \beta = \frac{v}{c}.$$

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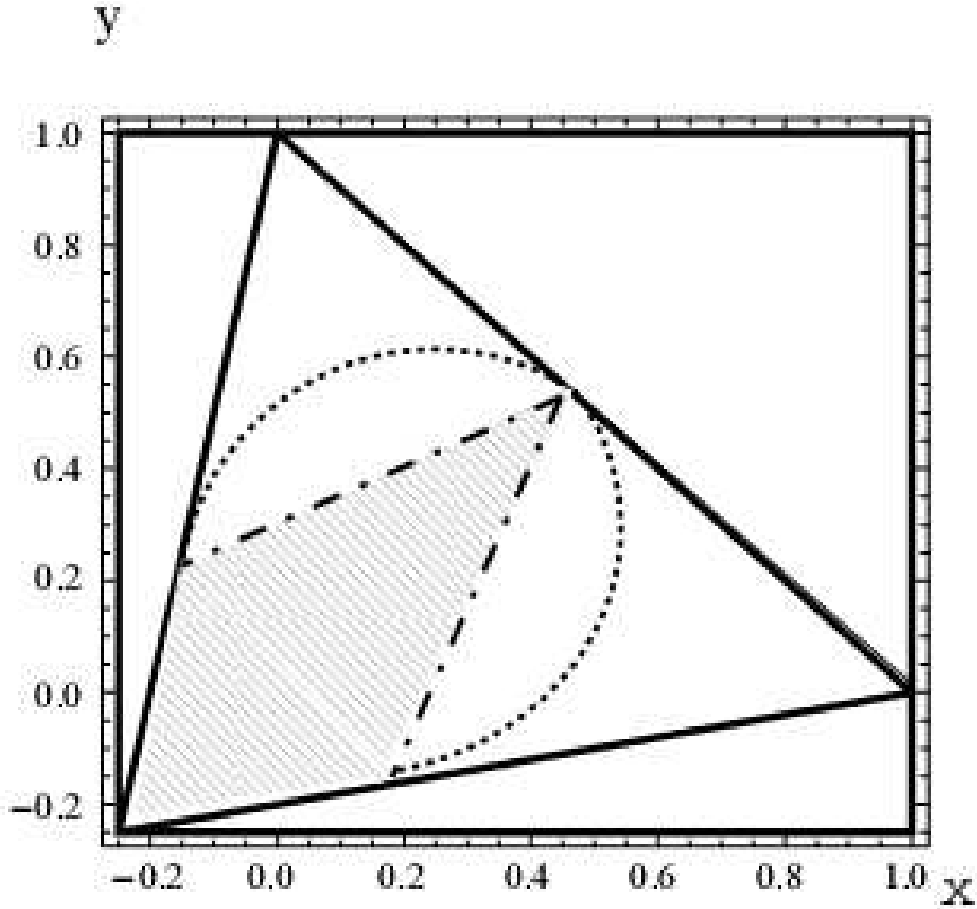


Figure 1.

Figure Caption

- Fig. 1: (Color online) Here, the space of the density matrices under Lorentz transformation is shown. The solid line (black triangle) shows the positivity condition. The hatched area with the dashed lines is the boundary of separability in rest frame(i.e., $\theta = 0$). The dotted black curve represents the boundary of separability density matrix under Lorentz transformation when $\theta = \frac{\pi}{4}$. For $\theta = \frac{\pi}{2}$ leads to black triangle and positivity condition in rest frame.

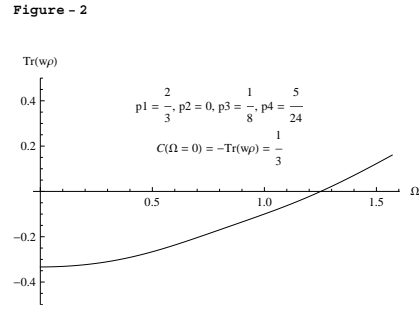


Figure 2.

Figure Caption

- Fig. 2: The $\text{Tr}[\mathcal{W}_{2 \otimes 2} \cdot \rho_{\frac{1}{2}}^\Lambda]$ versus Wigner angle Ω_p . Here $p_1 = \frac{2}{3}$, $p_2 = 0$, $p_3 = \frac{1}{8}$, and $p_4 = \frac{5}{24}$.